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Third-order integrable difference equations generated by a pair of second-order equations

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Abstract
We show that the third-order difference equations proposed by Hirota, Kimura and Yahagi are generated by a pair of second-order difference equations. In some cases, the pair of the second-order equations are equivalent to the Quispel–Robert–Thomson (QRT) system, but in the other cases, they are irrelevant to the QRT system. We also discuss an ultradiscretization of the equations.

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1. Introduction

Discrete integrable systems have attracted much attention and a lot of studies have been done from various points of view, such as integrability criteria (singularity confinement property [1], algebraic entropy [2]), geometric or algebraic description of the equations [3–8] and so on.

In particular, second-order integrable difference equations including Quispel–Robert–Thomson (QRT) system [9, 10] and discrete Painlevé equations [11], which are regarded as non-autonomous variations of QRT system, have been extensively studied, and a number of significant properties have been obtained.

For example, a symmetric version of QRT system is defined by the following form:

$$x_{n+1} = \frac{f_1(x_n) - x_{n-1} f_2(x_n)}{f_2(x_n) - x_{n-1} f_3(x_n)}, \tag{1}$$

where $f_j(x)$ is defined by

$$\begin{pmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{pmatrix} = A \begin{pmatrix} x^2 \\ x \\ 1 \end{pmatrix} \times B \begin{pmatrix} x^2 \\ x \\ 1 \end{pmatrix}, \tag{2}$$

with arbitrary symmetric 3×3 matrices A and B. This equation has a conserved quantity $h(x_{n-1}, x_n)$ defined by A and B. Moreover, a general solution is described by an elliptic function.

However, very few results have been obtained for third-order integrable difference equations. Research on such equations are important in order to reveal integrable structures of general discrete integrable systems.

In this paper, we investigate third-order integrable difference equations proposed by Hirota, Kimura and Yahagi [12] and show that they are generated by a pair of second-order integrable difference equations. Moreover, we also discuss their ultradiscretization.

Hirota, Kimura and Yahagi have investigated third-order difference equations of the form

$$x_{n+2}x_{n-1} = \frac{a_0 + a_1x_n + a_2x_{n+1} + a_3x_nx_{n+1}}{b_0 + b_1x_n + b_2x_{n+1} + b_3x_nx_{n+1}}, \tag{3}$$

and have found that nine equations,

$$x_{n+2}x_{n-1} = \frac{a_0 + a_1(x_n + x_{n+1}) + a_3x_nx_{n+1}}{a_3 + b_1(x_n + x_{n+1}) + b_3x_nx_{n+1}}, \tag{Y1}$$

$$x_{n+2}x_{n-1} = \frac{a_0(1 + x_n + x_{n+1}) + a_3x_nx_{n+1}}{a_0 + a_3(x_n + x_{n+1} + x_nx_{n+1})}, \tag{Y2}$$

$$x_{n+2}x_{n-1} = \frac{a_0(-1 + x_n - x_{n+1}) + a_3x_nx_{n+1}}{a_0 + a_3(x_n - x_{n+1} - x_nx_{n+1})}, \tag{Y3}$$

$$x_{n+2}x_{n-1} = \frac{a_0 + a_1(x_n + x_{n+1} + x_nx_{n+1})}{1 + x_n + x_{n+1} + x_nx_{n+1}}, \tag{Y4}$$

$$x_{n+2}x_{n-1} = \frac{a_1(x_n - x_{n+1}) + a_3x_nx_{n+1}}{a_3 + b_1(-x_n + x_{n+1})}, \tag{Y5}$$

$$x_{n+2}x_{n-1} = \frac{a_3x_nx_{n+1}}{b_1(x_n + x_{n+1}) + b_3x_nx_{n+1}}, \tag{Y6}$$

$$x_{n+2}x_{n-1} = \frac{a_0 + a_1x_n}{a_1x_n + a_0x_nx_{n+1}}, \tag{Y7}$$

$$x_{n+2}x_{n-1} = \frac{a_0 + a_1x_n}{-a_1x_n + a_0x_nx_{n+1}}, \tag{Y8}$$

$$x_{n+2}x_{n-1} = \frac{x_n + x_nx_{n+1}}{1 + x_n}, \tag{Y9}$$

are integrable in the sense that they have two independent conserved quantities. A remarkable property of these equations is that their trajectory of a solution in three-dimensional phase space looks like a composition of two separate curves. Figure 1 is an example of such trajectories in 3D phase space which is generated by

$$y_{n+2}y_{n-1} = a + y_n + y_{n+1}, \tag{4}$$

where the equation is obtained through a variable transformation $y_n = \frac{a_3}{b_1x_n}$, $a = \frac{a_3b_3}{b_1^2}$ from equation ($Y6$). Moreover, it is an important fact that odd step points belong to one curve and even step points belong to the other.

This fact strongly suggests that a combination of lower dimensional integrable equations determines the integrability of the third-order difference equation. We show that this is true for all nine equations in the following section.

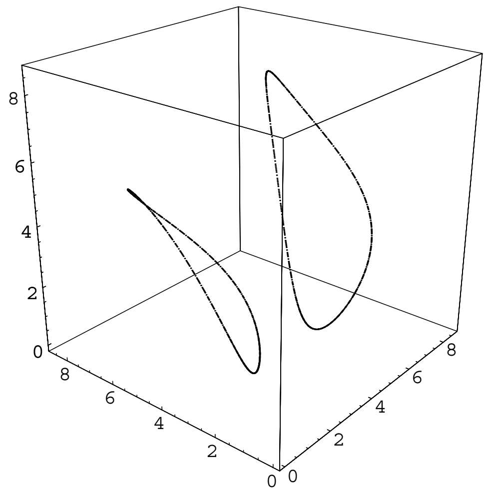

Figure 1. A trajectory of a solution to equation (4) for $a = 2.0$, $y_0 = 1.0$, $y_1 = 2.0$, $y_2 = 1.5$.

2. Pair of second-order integrable equations generating a third-order equation

2.1. Y6

If we take a backward difference of equation (4)

$$y_{n+2}y_{n-1} = a + y_n + y_{n+1},$$

we obtain

$$\Delta_n(y_{n+2}y_{n-1} - a - y_n - y_{n+1}) = 0, \tag{5}$$

where Δ_n is a difference operator defined by $\Delta_n f_n = f_n - f_{n-1}$. Equation (5) can be written as

$$\frac{(1 + y_{n+2})(1 + y_n)}{y_{n+1}} = \frac{(1 + y_n)(1 + y_{n-2})}{y_{n-1}}. \tag{6}$$

This formula means that there are constants which depend on the initial values and on a parity of n. Hence, we obtain

$$\begin{cases} \dfrac{(1 + g_{n+1})(1 + g_n)}{h_n} = c_0, \\ \dfrac{(1 + h_n)(1 + h_{n-1})}{g_n} = c_1, \end{cases} \tag{7}$$

where $g_n = y_{2n}$, $h_n = y_{2n+1}$, $c_0 = \frac{(1+y_0)(1+y_2)}{y_1}$, $c_1 = \frac{(1+y_1)(1+y_3)}{y_2}$. From equations (4) and (7), we obtain a pair of second-order difference equations

$$g_{n+1} = \frac{(1 + ac_0) + (1 + c_0)g_n}{g_{n-1}(1 + g_n)}, \tag{8}$$

$$h_{n+1} = \frac{(1 + ac_1) + (1 + c_1)h_n}{h_{n-1}(1 + h_n)}, \tag{9}$$

where equation (8) is a equation for even steps and equation (9) is a equation for odd steps, respectively. Equations (8) and (9) can be written in the QRT form

$$\begin{cases} g_{n+1} = \dfrac{G_1(g_n) - g_{n-1}G_2(g_n)}{G_2(g_n) - g_{n-1}G_3(g_n)}, \\ h_{n+1} = \dfrac{H_1(h_n) - h_{n-1}H_2(h_n)}{H_2(h_n) - h_{n-1}H_3(h_n)}, \end{cases} \tag{10}$$

where

$$A(c) = \begin{pmatrix} 1 & 2+c & 1+c \\ 2+c & 0 & 2+2c+ac+c^2 \\ 1+c & 2+2c+ac+c^2 & (1+c)(1+ac) \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{11}$$

$$\begin{pmatrix} G_1(x) \\ G_2(x) \\ G_3(x) \end{pmatrix} = A(c_0) \begin{pmatrix} x^2 \\ x \\ 1 \end{pmatrix} \times B \begin{pmatrix} x^2 \\ x \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} H_1(x) \\ H_2(x) \\ H_3(x) \end{pmatrix} = A(c_1) \begin{pmatrix} x^2 \\ x \\ 1 \end{pmatrix} \times B \begin{pmatrix} x^2 \\ x \\ 1 \end{pmatrix}. \tag{12}$$

Consequently, conserved quantities of equations (8) and (9) are given as

$$k_0 = \big(g_n^2 g_{n+1}^2 + (2+c_0)(g_n + g_{n+1})g_n g_{n+1} + (1+c_0)\big(g_n^2 + g_{n+1}^2\big) \\ + \big(2 + 2c_0 + ac_0 + c_0^2\big)(g_n + g_{n+1}) + (1+c_0)(1+ac_0)\big)\big/(g_n g_{n+1}), \tag{13}$$

and

$$k_1 = \big(h_n^2 h_{n+1}^2 + (2+c_1)(h_n + h_{n+1})h_n h_{n+1} + (1+c_1)\big(h_n^2 + h_{n+1}^2\big) \\ + \big(2 + 2c_1 + ac_1 + c_1^2\big)(h_n + h_{n+1}) + (1+c_1)(1+ac_1)\big)\big/(h_n h_{n+1}). \tag{14}$$

Hence, invariant curves of equations (8) and (9) are given by the above equations. These curves determine the structure of the trajectory of a solution to equation ($Y6$) in 3D phase space as shown in figure 1. This is the simplified integrability structure of equation ($Y6$) and we show below that a similar structure exists in the other eight equations.

2.2. *Y1*

From equation ($Y1$)

$$x_{n+2}x_{n-1} = \frac{a_0 + a_1(x_n + x_{n+1}) + a_3 x_n x_{n+1}}{a_3 + b_1(x_n + x_{n+1}) + b_3 x_n x_{n+1}}, \tag{Y1}$$

we obtain

$$\Delta_n((a_3 + b_1(x_n + x_{n+1}) + b_3 x_n x_{n+1})x_{n+2}x_{n-1} - (a_0 + a_1(x_n + x_{n+1}) + a_3 x_n x_{n+1})) = 0. \tag{15}$$

Equation (15) can be written as

$$b_3 x_{n+2} x_n + b_1 \left(x_{n+2} + x_n + \frac{x_{n+2}x_n}{x_{n+1}} \right) + a_3 \left(\frac{x_{n+2}}{x_{n+1}} + \frac{x_n}{x_{n+1}} \right) + \frac{a_1}{x_{n+1}} \\ = b_3 x_n x_{n-2} + b_1 \left(x_n + x_{n-2} + \frac{x_n x_{n-2}}{x_{n-1}} \right) + a_3 \left(\frac{x_n}{x_{n-1}} + \frac{x_{n-2}}{x_{n-1}} \right) + \frac{a_1}{x_{n-1}}. \tag{16}$$

Hence, we obtain

$$\begin{cases} g_n = \dfrac{a_1 + a_3(h_{n-1} + h_n) + b_1 h_{n-1} h_n}{c_1 - b_1(h_{n-1} + h_n) - b_3 h_{n-1} h_n}, \\ h_n = \dfrac{a_1 + a_3(g_n + g_{n+1}) + b_1 g_n g_{n+1}}{c_0 - b_1(g_n + g_{n+1}) - b_3 g_n g_{n+1}}, \end{cases} \tag{17}$$

where $g_n = x_{2n}, h_n = x_{2n+1}$ and

$$c_0 = \frac{1}{x_1}(b_1x_0x_2 + a_3x_2 + a_3x_0 + a_1) + b_3x_0x_2 + b_1(x_0 + x_2),$$
$$c_1 = \frac{1}{x_2}(b_1x_1x_3 + a_3x_3 + a_3x_1 + a_1) + b_3x_1x_3 + b_1(x_1 + x_3). \tag{18}$$

From equations ($Y1$) and (17), we obtain a pair of QRT systems

$$\begin{cases} g_{n+1} = \dfrac{G_1(g_n) - g_{n-1}G_2(g_n)}{G_2(g_n) - g_{n-1}G_3(g_n)}, \\ h_{n+1} = \dfrac{H_1(h_n) - h_{n-1}H_2(h_n)}{H_2(h_n) - h_{n-1}H_3(h_n)}, \end{cases} \tag{19}$$

where

$$A(c) = \begin{pmatrix} b_1^2 - 2a_3b_3 + \dfrac{a_1b_3^2}{b_1} & 0 & a_3^2 - a_1b_1 \\ 0 & 2a_3^2 - a_0b_3 + 2a_3c - \dfrac{a_1b_3c}{b_1} & 2a_1a_3 - a_0b_1 + a_1c \\ a_3^2 - a_1b_1 & 2a_1a_3 - a_0b_1 + a_1c & a_1^2 + a_0c \end{pmatrix}, \tag{20}$$

$$B(c) = \begin{pmatrix} b_3 & b_1 & 0 \\ b_1 & -c & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{21}$$

$$\begin{pmatrix} G_1(x) \\ G_2(x) \\ G_3(x) \end{pmatrix} = A(c_0)\begin{pmatrix} x^2 \\ x \\ 1 \end{pmatrix} \times B(c_0)\begin{pmatrix} x^2 \\ x \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} H_1(x) \\ H_2(x) \\ H_3(x) \end{pmatrix} = A(c_1)\begin{pmatrix} x^2 \\ x \\ 1 \end{pmatrix} \times B(c_1)\begin{pmatrix} x^2 \\ x \\ 1 \end{pmatrix}. \tag{22}$$

2.3. *Y4*

From equation ($Y4$)

$$x_{n+2}x_{n-1} = \frac{a_0 + a_1(x_{n+1} + x_n + x_{n+1}x_n)}{1 + x_n + x_{n+1} + x_{n+1}x_n}, \tag{$Y4$}$$

we obtain

$$\Delta_n((1 + x_n + x_{n+1} + x_{n+1}x_n)x_{n+2}x_{n-1} - (a_0 + a_1(x_{n+1} + x_n + x_{n+1}x_n))) = 0. \tag{23}$$

Equation (23) can be written as

$$(x_{n+2} + a_1)(x_n + a_1)\frac{x_{n+1} + 1}{x_{n+1}} = (x_n + a_1)(x_{n-2} + a_1)\frac{x_{n-1} + 1}{x_{n-1}}. \tag{24}$$

From this equation, we obtain

$$\begin{cases} g_n = \dfrac{(h_n + a_1)(h_{n-1} + a_1)}{c_1 - (h_n + a_1)(h_{n-1} + a_1)}, \\ h_n = \dfrac{(g_{n+1} + a_1)(g_n + a_1)}{c_0 - (g_{n+1} + a_1)(g_n + a_1)}, \end{cases} \tag{25}$$

where $g_n = x_{2n}, h_n = x_{2n+1}$ and

$$c_0 = (x_2 + a_1)(x_0 + a_1)\frac{x_1 + 1}{x_1}, \tag{26}$$

$$c_1 = (x_3 + a_1)(x_1 + a_1)\frac{x_2 + 1}{x_2}. \tag{27}$$

From equations ($Y4$) and (25), we obtain a pair of QRT systems

$$\begin{cases} g_{n+1} = \dfrac{G_1(g_n) - g_{n-1}G_2(g_n)}{G_2(g_n) - g_{n-1}G_3(g_n)}, \\ h_{n+1} = \dfrac{H_1(h_n) - h_{n-1}H_2(h_n)}{H_2(h_n) - h_{n-1}H_3(h_n)}, \end{cases} \tag{28}$$

where

$$A(c) = \begin{pmatrix} (a_0 - a_1 - c)(a_1 - 1) & 0 & -(a_0 - a_1 - c)(a_1 - 1)a_1^2 \\ 0 & a_{22} & a_{23} \\ -(a_0 - a_1 - c)(a_1 - 1)a_1^2 & a_{32} & a_{33} \end{pmatrix},$$

$$\begin{aligned} a_{22} &= -2(a_0 - a_1)(a_1 - 1)a_1^2 + c\left(a_0 - 2a_1 + a_1^3 - c\right), \\ a_{23} &= -2(a_0 - a_1)(a_1 - 1)a_1^3 + ca_1\left(-a_0 + 2a_0a_1 - 2a_1^2 + a_1^3 - a_1c\right), \\ a_{32} &= a_{23}, \\ a_{33} &= -(a_0 - a_1)(a_1 - 1)a_1^4 + ca_1\left(-a_0a_1 + 2a_0a_1^2 - a_1^3 - a_0c\right) \end{aligned} \tag{29}$$

$$B(c) = \begin{pmatrix} 1 & a_1 & 0 \\ a_1 & a_1^2 - c & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{30}$$

$$\begin{pmatrix} G_1(x) \\ G_2(x) \\ G_3(x) \end{pmatrix} = A(c_0)\begin{pmatrix} x^2 \\ x \\ 1 \end{pmatrix} \times B(c_0)\begin{pmatrix} x^2 \\ x \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} H_1(x) \\ H_2(x) \\ H_3(x) \end{pmatrix} = A(c_1)\begin{pmatrix} x^2 \\ x \\ 1 \end{pmatrix} \times B(c_1)\begin{pmatrix} x^2 \\ x \\ 1 \end{pmatrix}. \tag{31}$$

2.4. Y5

From equation ($Y5$)

$$x_{n+2}x_{n-1} = \frac{a_1x_n - a_1x_{n+1} + a_3x_nx_{n+1}}{a_3 - b_1x_n + b_1x_{n+1}}, \tag{$Y5$}$$

we obtain

$$\begin{aligned} &x_{n+2}x_{n-1}(a_3 - b_1x_n + b_1x_{n+1}) - (a_1x_n - a_1x_{n+1} + a_3x_nx_{n+1}) \\ &\quad + x_{n+1}x_{n-2}(a_3 - b_1x_{n-1} + b_1x_n) - (a_1x_{n-1} - a_1x_n + a_3x_{n-1}x_n) = 0. \end{aligned} \tag{32}$$

Note that we do not take a backward difference but a sum of equation ($Y5$) here. Equation (32) can be written as

$$\begin{aligned} &a_3\frac{x_{n+2} - x_n}{x_{n+1}} - \frac{a_1}{x_{n+1}} + b_1(x_{n+2} + x_n) - b_1\frac{x_{n+2}x_n}{x_{n+1}} \\ &\quad = a_3\frac{x_n - x_{n-2}}{x_{n-1}} - \frac{a_1}{x_{n-1}} + b_1(x_n + x_{n-2}) - b_1\frac{x_nx_{n-2}}{x_{n-1}}. \end{aligned} \tag{33}$$

From this equation, we obtain

$$\begin{cases} g_n = \dfrac{a_3(h_n - h_{n-1}) - a_1 - b_1h_nh_{n-1}}{c_1 - b_1(h_n + h_{n-1})}, \\ h_n = \dfrac{a_3(g_{n+1} - g_n) - a_1 - b_1g_{n+1}g_n}{c_0 - b_1(g_{n+1} + g_n)}, \end{cases} \tag{34}$$

where $g_n = x_{2n}, h_n = x_{2n+1}$ and

$$c_0 = a_3 \frac{x_2 - x_0}{x_1} - \frac{a_1}{x_1} + b_1(x_2 + x_0) - b_1 \frac{x_2 x_0}{x_1}, \tag{35}$$

$$c_1 = a_3 \frac{x_3 - x_1}{x_2} - \frac{a_1}{x_2} + b_1(x_3 + x_1) - b_1 \frac{x_3 x_1}{x_2}. \tag{36}$$

From equations ($Y5$) and (34), we obtain a pair of QRT systems

$$\begin{cases} g_{n+1} = \dfrac{G_1(g_n) - g_{n-1}G_2(g_n)}{G_2(g_n) - g_{n-1}G_3(g_n)}, \\ h_{n+1} = \dfrac{H_1(h_n) - h_{n-1}H_2(h_n)}{H_2(h_n) - h_{n-1}H_3(h_n)}, \end{cases} \tag{37}$$

where

$$A(c) = \begin{pmatrix} b_1^2 & 0 & -a_3^2 - a_1 b_1 \\ 0 & 2a_3^2 & a_1 c \\ -a_3^2 - a_1 b_1 & a_1 c & a_1^2 \end{pmatrix}, \tag{38}$$

$$B(c) = \begin{pmatrix} 0 & b_1 & 0 \\ b_1 & -c & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{39}$$

$$\begin{pmatrix} G_1(x) \\ G_2(x) \\ G_3(x) \end{pmatrix} = A(c_0) \begin{pmatrix} x^2 \\ x \\ 1 \end{pmatrix} \times B(c_0) \begin{pmatrix} x^2 \\ x \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} H_1(x) \\ H_2(x) \\ H_3(x) \end{pmatrix} = A(c_1) \begin{pmatrix} x^2 \\ x \\ 1 \end{pmatrix} \times B(c_1) \begin{pmatrix} x^2 \\ x \\ 1 \end{pmatrix}. \tag{40}$$

2.5. *Y7*

Introducing a variable transformation $x_n = \frac{f_{n+1}}{f_n}$ to equation ($Y7$)

$$x_{n+2} x_{n-1} = \frac{a_0 + a_1 x_n}{a_1 x_n + a_0 x_n x_{n+1}}, \tag{Y7}$$

we obtain

$$a_0(f_{n+1} + f_{n-1}) + a_1 \frac{f_{n+1} f_{n-1}}{f_n} = a_0(f_{n+3} + f_{n+1}) + a_1 \frac{f_{n+3} f_{n+1}}{f_{n+2}}. \tag{41}$$

From this equation, we obtain

$$\begin{cases} a_0(g_{n+1} + g_n) + a_1 \dfrac{g_{n+1} g_n}{h_n} = c_0 \\ a_0(h_n + h_{n-1}) + a_1 \dfrac{h_n h_{n-1}}{g_n} = c_1 \end{cases} \tag{42}$$

where $g_n = f_{2n}, h_n = f_{2n+1}$ and

$$c_0 = a_0(g_1 + g_0) + a_1 \frac{g_1 g_0}{h_0} = f_0 a_0 \left(1 + x_0 x_1 + \frac{a_1}{a_0} x_1\right), \tag{43}$$

$$c_1 = a_0(h_1 + h_0) + a_1 \frac{h_1 h_0}{g_1} = f_0 a_0 x_0 \left(1 + x_1 x_2 + \frac{a_1}{a_0} x_2\right). \tag{44}$$

From equation (42), we obtain a pair of QRT system

$$\begin{cases} g_{n+1} = \dfrac{G_1(g_n) - g_{n-1}G_2(g_n)}{G_2(g_n) - g_{n-1}G_3(g_n)}, \\ h_{n+1} = \dfrac{H_1(h_n) - h_{n-1}H_2(h_n)}{H_2(h_n) - h_{n-1}H_3(h_n)}, \end{cases} \tag{45}$$

where

$$\begin{pmatrix} G_1(x) \\ G_2(x) \\ G_3(x) \end{pmatrix} = \begin{pmatrix} 0 & 0 & a_0^2(a_0^2 - a_1^2) \\ 0 & (a_0^2 - a_1^2)(2a_0^2 - a_1^2) & -a_0(2a_0^2c_0 - a_1^2c_0 + a_0a_1c_1) \\ a_0^2(a_0^2 - a_1^2) & -a_0(2a_0^2c_0 - a_1^2c_0 + a_0a_1c_1) & a_0c_0(a_0c_0 + a_1c_1) \end{pmatrix}$$

$$\times \begin{pmatrix} x^2 \\ x \\ 1 \end{pmatrix} \begin{pmatrix} a_0a_1 & a_0c_1 & 0 \\ a_0c_1 & -c_0c_1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x^2 \\ x \\ 1 \end{pmatrix}, \tag{46}$$

$$\begin{pmatrix} H_1(x) \\ H_2(x) \\ H_3(x) \end{pmatrix} = \begin{pmatrix} 0 & 0 & a_0^2(a_0^2 - a_1^2) \\ 0 & (a_0^2 - a_1^2)(2a_0^2 - a_1^2) & -a_0(2a_0^2c_1 - a_1^2c_1 + a_0a_1c_0) \\ a_0^2(a_0^2 - a_1^2) & -a_0(2a_0^2c_1 - a_1^2c_1 + a_0a_1c_0) & a_0c_1(a_0c_1 + a_1c_0) \end{pmatrix}$$

$$\times \begin{pmatrix} x^2 \\ x \\ 1 \end{pmatrix} \begin{pmatrix} a_0a_1 & a_0c_0 & 0 \\ a_0c_0 & -c_0c_1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x^2 \\ x \\ 1 \end{pmatrix}. \tag{47}$$

2.6. Y8

Introducing a dependent variable transformation $x_n = \mathrm{i}y_n$, $a_1' = \mathrm{i}a_1$ to equation ($Y8$)

$$x_{n+2}x_{n-1} = \frac{a_0 + a_1x_n}{-a_1x_n + a_0x_nx_{n+1}}, \tag{$Y8$}$$

we obtain

$$y_{n+2}y_{n-1} = \frac{a_0 + a_1'y_n}{a_1'y_n + a_0y_ny_{n+1}}.$$

This is equivalent to equation ($Y7$).

2.7. Y9

Introducing a variable transformation $x_n = \frac{\tilde{f}_{n+1}}{\tilde{f}_n}$ to equation ($Y9$)

$$x_{n+2}x_{n-1} = \frac{x_n + x_nx_{n+1}}{1 + x_n}, \tag{$Y9$}$$

we obtain

$$\frac{\tilde{f}_{n+3}\tilde{f}_n}{\tilde{f}_{n+2} + \tilde{f}_{n+1}} = \frac{\tilde{f}_{n+2}\tilde{f}_{n-1}}{\tilde{f}_{n+1} + \tilde{f}_n}. \tag{48}$$

From this equation, we obtain

$$\tilde{f}_{n+2}\tilde{f}_{n-1} = \alpha(\tilde{f}_{n+1} + \tilde{f}_n). \tag{49}$$

By scaling as $\tilde{f}_n = \alpha f_n$, we obtain

$$f_{n+2}f_{n-1} = f_{n+1} + f_n. \tag{50}$$

This is equation (4) in the case of $a = 0$.

2.8. Y2

Equation ($Y2$) is written as

$$x_{n+2}x_{n-1} = \frac{a + ax_n + ax_{n+1} + x_n x_{n+1}}{a + x_n + x_{n+1} + x_n x_{n+1}} \tag{Y2}$$

where $a = a_0/a_3$. Equation ($Y2$) is generated by a pair of second-order equation

$$\begin{cases} g_{n+1} = -g_n\left(1 + \dfrac{(a+g_{n-1})\big(b_0\big(a-g_n^2\big)-ac\big)}{(a-1)^2(a-g_n)(g_n+g_{n-1}) + b_0 g_n(a+ag_n+ag_{n-1}+g_n g_{n-1}) - acg_n}\right), \\ h_{n+1} = -h_n\left(1 + \dfrac{(a+h_{n-1})\big(b_1\big(a-h_n^2\big)-ac\big)}{(a-1)^2(a-h_n)(h_n+h_{n-1}) + b_1 g_n(a+ah_n+ah_{n-1}+h_n h_{n-1}) - ach_n}\right), \end{cases} \tag{51}$$

where $g_n = x_{2n}, h_n = x_{2n+1}$ and

$$c = \frac{1}{x_0x_1x_2}(1+x_0)(1+x_1)(1+x_2)(a(1+x_0+x_1+x_2)+x_0x_1+x_1x_2+x_2x_0+x_0x_1x_2), \tag{52}$$

$$b_0 = \frac{1}{x_0x_1x_2}\{(a-1)x_0x_2(1+x_1)^2 + (1+x_1+x_0x_1+x_1x_2)(a(1+x_0+x_1+x_2) + x_0x_1+x_1x_2+x_2x_0+x_0x_1x_2), \tag{53}$$

$$b_1 = \frac{1}{x_1x_2x_3}\{(a-1)x_1x_3(1+x_2)^2 + (1+x_2+x_1x_2+x_2x_3)(a(1+x_1+x_2+x_3) + x_1x_2+x_2x_3+x_3x_1+x_1x_2x_3). \tag{54}$$

Equation (51) does not belong to the QRT system as it stands, though there is still a possibility that it could be reduced to the system through a change of variable.

2.9. Y3

For equation ($Y3$), we can only show numerical results. For various parameters a_0, a_3 and initial values $x_0 \sim x_2$, it is generated by a pair of second-order equations on even steps (g_n) and odd steps (h_n). In any case, both equations follows

$$g_{n+1} = \frac{\sum_{0\leqslant i,j\leqslant 3} a_{ij} g_{n-1}^i g_n^j}{\sum_{0\leqslant i,j\leqslant 3} b_{ij} g_{n-1}^i g_n^j}, \qquad h_{n+1} = \frac{\sum_{0\leqslant i,j\leqslant 3} c_{ij} h_{n-1}^i h_n^j}{\sum_{0\leqslant i,j\leqslant 3} d_{ij} h_{n-1}^i h_n^j}, \tag{55}$$

where $a_{ij} \sim d_{ij}$ are constant obtained from parameters and initial values numerically. This fact strongly suggests that equation ($Y3$) is also derived from a pair of second-order equations.

3. Ultradiscretization

In this section, we consider an ultradiscrete version of the third-order integrable equations [13, 14]. Since ultradiscretization requires a positivity of parameters and dependent variables of the equations, ($Y1$), ($Y2$), ($Y4$), ($Y6$), ($Y7$) and ($Y9$) are ultradiscretizable. We show that the procedure of ultradiscretization works well by equation (4) as an example.

If we use transformations $y_n = \exp\left(\frac{Y_n}{\epsilon}\right)$, $a = \exp\left(\frac{A}{\epsilon}\right)$ for equation (4)

$$y_{n+2}y_{n-1} = a + y_n + y_{n+1},$$

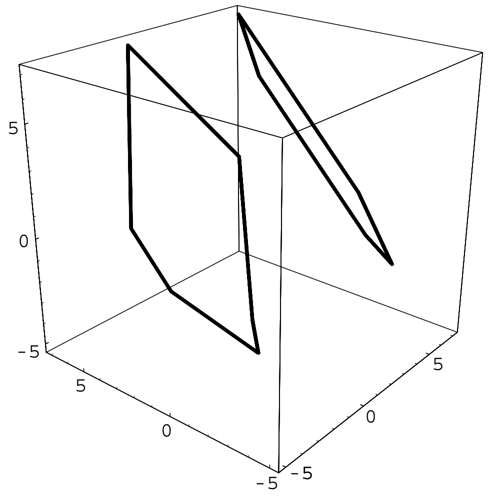

Figure 2. A trajectory of a solution to equation (56) for $A = 2.0$, $Y_0 = 2.0$, $Y_1 = 2.01$, $Y_2 = -5.1$.

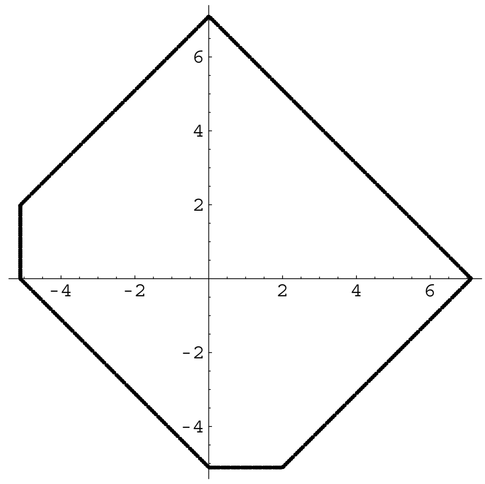

Figure 3. An invariant curve of equation (57) for $A = 2.0$, $Y_0 = 2.0$, $Y_1 = 2.01$, $Y_2 = -5.1$.

and take a limit $\epsilon \to +0$, then we have

$$Y_{n+2} = \max(A, Y_n, Y_{n+1}) - Y_{n-1}. \tag{56}$$

This is an ultradiscrete version of equation (4). Figure 2 shows a trajectory of a solution to equation (56) in 3D phase space.

It follows from the result for equation (4) in the previous section that equation (56) is generated by a pair of ultradiscrete QRT system

$$U_{n+1} = \max(0, A + C_0, U_n + \max(0, C_0)) - U_{n-1} - \max(0, U_n), \tag{57}$$

$$V_{n+1} = \max(0, A + C_1, V_n + \max(0, C_1)) - V_{n-1} - \max(0, V_n), \tag{58}$$

$$C_0 = \max(0, Y_0) + \max(0, Y_2) - Y_1, \tag{59}$$

$$C_1 = \max(0, Y_1) + \max(0, Y_3) - Y_2, \tag{60}$$

and invariant curves become

$$\max(U_n + U_{n+1}, \max(U_n, U_{n+1}) + \max(0, C_0), \max(U_n - U_{n+1}, U_{n+1} - U_n) + \max(0, C_0),$$
$$\max(-U_n, -U_{n+1}) + \max(0, C_0, A + C_0, 2C_0),$$
$$-U_n - U_{n+1} + \max(0, C_0) + \max(0, A + C_0)) = K_0, \tag{61}$$

and

$$\max(V_n + V_{n+1}, \max(V_n, V_{n+1}) + \max(0, C_1), \max(V_n - V_{n+1}, V_{n+1} - V_n) + \max(0, C_1),$$
$$\max(-V_n, -V_{n+1}) + \max(0, C_1, A + C_1, 2C_1),$$
$$-V_n - V_{n+1} + \max(0, C_1) + \max(0, A + C_1)) = K_1. \tag{62}$$

Figure 3 shows invariant curves for equation (57) determined by equation (61).

4. Conclusion

In this paper, we have shown that the third-order integrable difference equations proposed by Hirota, Kimura and Yahagi are generated by a pair of second-order integrable difference equations. In the case of equations ($Y1$) and ($Y4$)–($Y9$), second-order difference equations are a special case of the QRT system. In the case of equations ($Y2$) and ($Y3$), second-order equations may not be the QRT system. Furthermore, we have shown that the procedure of ultradiscretization works well for third-order equation, and derived second-order equations and invariants curves are also ultradiscretizable.

Although the whole integrability structure of the general third-order equations is still unknown, our work could be one of the keys to understanding the structure. Generating our results, that is, investigating a connection between the general QRT system and the third-order equations is an important future problem.

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